

A NOTE ON THE PROBABILISTIC N-BANACH SPACES

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ABSTRACT. In this paper, we define probabilistic n -Banach spaces along with some concepts in this field and study convergence in these spaces by some lemmas and theorem.

1. INTRODUCTION AND PRELIMINARIES

In [7] K. Menger introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function instead of nonnegative real numbers as values of the metric. The concept of probabilistic normed spaces were introduced by Šerstnev in [12]. New definition of probabilistic normed spaces and some concepts in this field were studied by Alsina, Schweizer, Sklar, Pourmoslemi and Salimi [1, 2, 3, 8, 9]. It corresponds to the situations when we do not know exactly the distance between two points, we know only probabilities of possible values of this

2000 Mathematics Subject Classification. Primary 54E70; Secondary 46S50.

Key words and phrases. n -normed space, probabilistic N -Banach spaces.

distance. The probabilistic generalization of metric spaces appears to be well adapted for the investigation of quantum particle physics particularly in connections with both string and ε^∞ theory which were given and studied by El Naschie [5, 6]. A *distribution function (briefly a d.f.)* is a function F from the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ and the subset of those d.f.'s such that $F(0) = 0$, will be denoted by Δ^+ and $D^+ \subseteq \Delta^+$ is defined as follows:

$$D^+ = \{F \in \Delta^+ : l^-F(+\infty) = 1\},$$

where $l^-f(x)$ denotes the left limit of the function f at the point x . By setting $F \leq G$ whenever $F(x) \leq G(x)$ for all x in \mathbb{R} , the maximal element for Δ^+ in this order is the d.f. given by

$$\mathcal{H}_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

The space Δ can be metrized in several equivalent ways [10, 11, 13, 14] in such a manner that the metric topology coincides with the topology of weak convergence for distribution functions. Here, we assume that Δ is metrized by the Sibley metric d_s . If F and G are d.f.'s and h is in $]0, 1]$, let $(F, G; h)$ denote the condition

$$F(x - h) - h \leq G(x) \leq F(x + h) + h, \quad x \in]-\frac{1}{h}, \frac{1}{h}].$$

Then the Sibley metric is defined by

$$d_s(F, G) := \inf\{h \in]0, 1] : \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

A *t-norm* T is a two-place function $T : I \times I \rightarrow I$ which is associative, commutative, nondecreasing in each place and such that $T(a, 1) = a$, for all $a \in [0, 1]$. A *triangle function* is a binary operation on Δ^+ , namely

a function $\tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+$ that is associative, commutative, nondecreasing and which has \mathcal{H}_0 as unit. That is, for all $F, G, H \in \Delta^+$, we have

$$\begin{aligned}\tau(\tau(F, G), H) &= \tau(F, \tau(G, H)), \\ \tau(F, G) &= \tau(G, F), \\ F \leq G &\implies \tau(F, H) \leq \tau(G, H), \\ \tau(F, \varepsilon_0) &= F.\end{aligned}$$

Continuity of a triangle functions means continuity with respect to the topology of weak convergence in Δ^+ . Typical continuous triangle functions are

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

$$\tau_{T^*}(F, G)(x) = \inf_{s+t=x} T^*(F(s), G(t)).$$

Here T is a continuous t-norm, i.e. a continuous binary operation on $[0, 1]$ that is commutative, associative, nondecreasing in each variable and has 1 as identity, and T^* is a continuous t-conorm, namely a continuous binary operation on $[0, 1]$ which is related to the continuous t-norm T through

$$T^*(x, y) = 1 - T(1 - x, 1 - y).$$

Definition 1.1. A *probabilistic normed space* is a quadruple (V, ν, τ, τ^*) , where V is a real vector space, τ and τ^* are continuous triangle functions, and ν is a mapping from V into Δ^+ such that, for all p, q in V , the following conditions hold:

- (PN1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$, where θ is the null vector in V ;
- (PN2) $\nu_{-p} = \nu_p$, for each $p \in V$;

(PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$, for all $p, q \in V$;

(PN4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, for all α in $[0, 1]$.

If the inequality (PN4) is replaced by the equality

$$\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p}),$$

then the probabilistic normed space is called *Šerstnev space* and, as a consequence, a condition stronger than (PN2) holds, namely

$$\nu_{\lambda p}(x) = \nu_p\left(\frac{x}{|\lambda|}\right),$$

for all $p \in V$, $\lambda \neq 0$ and $x \in \mathbb{R}$.

Definition 1.2. [4] Let L be a real linear space with $\dim L \geq n$ and $\|\cdot, \dots, \cdot\| : L^n \rightarrow \mathbb{R}$ a function. Then $(L, \|\cdot, \dots, \cdot\|)$ is called a linear n -normed space if

(n-N1) $\|x_1, x_2, \dots, x_n\| = 0$ if, and only if x_1, x_2, \dots, x_n are linearly dependents;

(n-N2) $\|x_1, x_2, \dots, x_n\| = \|x_{i_1}, x_{i_2}, \dots, x_{i_n}\|$ for every permutation (i_1, \dots, i_n) of $(1, \dots, n)$;

(n-N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$;

(n-N4) $\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$,

for all $\alpha \in \mathbb{R}$ and all $x, y, x_1, \dots, x_n \in L$. The function $\|\cdot, \dots, \cdot\|$ is called the n -norm on L .

2. MAIN RESULTS

Definition 2.1. Let L be a linear space with $\dim L \geq n$, τ a triangle function, and let \mathcal{F} be a mapping from L^n into Δ^+ . If the following conditions are satisfied :

(Pn-N1) $F_{x_1, x_2, \dots, x_n} = \mathcal{H}_0$ if x_1, x_2, \dots, x_n are linearly dependent;

- (Pn-N2) $F_{x_1, x_2, \dots, x_n} \neq \mathcal{H}_0$ if x_1, x_2, \dots, x_n are linearly independent;
- (Pn-N3) $F_{x_1, x_2, \dots, x_n} = F_{x_{j_1}, x_{j_2}, \dots, x_{j_n}}$, for every permutation (j_1, \dots, j_n) ;
- (Pn-N4) $F_{\alpha x_1, x_2, \dots, x_n}(t) = F_{x_1, x_2, \dots, x_n}(\frac{t}{|\alpha|})$, for every $t > 0, \alpha \neq 0$ and $x_1, x_2, \dots, x_n \in L$;
- (Pn-N5) $F_{x+y, x_2, \dots, x_n} \geq \tau(F_{x, x_2, \dots, x_n}, F_{y, x_2, \dots, x_n})$, whenever $x, y, x_2, \dots, x_n \in L$,

then \mathcal{F} is called a *probabilistic n -norm* on L and the triple (L, \mathcal{F}, τ) is called a *probabilistic n -normed space* (Briefly *P-nNspace*).

Suppose that L be a vector space with dimension d , where $2 \leq d < \infty$, unless otherwise stated. Fix $\{u_1, \dots, u_d\}$ to be a basis for L . Then we have the following:

Lemma 2.2. *Let (L, \mathcal{F}, τ) with continuous τ be a P-nNspace. A sequence $\{x_m\}$ in L is convergent to x in L if, and only if*

$$\lim_{m \rightarrow \infty} F_{x_m - x, y_2, \dots, y_{n-1}, u_i} = \mathcal{H}_0, \text{ for every } y_2, \dots, y_{n-1} \in L, i = 1, \dots, d.$$

Following lemma 2.2, we have:

Lemma 2.3. *Let (L, \mathcal{F}, τ) with continuous τ be a P-nN space. A sequence $\{x_m\}$ in L is convergent to x in L if, and only if*

$$\lim_{m \rightarrow \infty} \max\{F_{x_m - x, y_2, \dots, y_{n-1}, u_i} : y_2, \dots, y_{n-1} \in L, i = 1, \dots, d\} = \mathcal{H}_0.$$

Now with respect to the base is $\{u_1, \dots, u_d\}$ we can define a norm on L which we shall denote it by \mathcal{F}_x^∞ as follows:

$$\mathcal{F}_x^\infty := \max\{F_{x, y_2, \dots, y_{n-1}, u_i} : y_2, \dots, y_{n-1} \in L, i = 1, \dots, d\}.$$

In fact it's not difficult to see:

- 1) $\mathcal{F}_x^\infty = \mathcal{H}_0$ if, and only if $x = 0$,

- 2) $F_{\alpha x}^\infty(t) = F_x^\infty(\frac{t}{|\alpha|})$,
 3) $F_{x+y}^\infty \geq \tau(F_x^\infty, F_y^\infty)$, for $x, y \in L$, $\alpha \neq 0$, $t > 0$.

Note that if we choose another basis for L , say $\{v_1, \dots, v_d\}$, and define the norm \mathcal{F}^∞ with respect to it, then the resulting norm will be equivalent to the one defined with respect to $\{u_1, \dots, u_d\}$.

Using the \mathcal{F}^∞ in lemma 2.3 we can write it as follows:

Lemma 2.4. *A sequence $\{x_m\}$ in L is convergent to x in L if, and only if $\lim_{m \rightarrow \infty} F_{x_m - x}^\infty = \mathcal{H}_0$.*

Also associated to the derived norm \mathcal{F}^∞ , we can define the open balls $B_{\{u_1, \dots, u_d\}}(x, t)$ centered at x having radius t by

$$B_{\{u_1, \dots, u_d\}}(x, t) := \{y \in L : F_{x-y}^\infty(t) > 1 - t\}.$$

With respect to above balls, lemma 2.4 be comes:

Lemma 2.5. *A sequence $\{x_m\}$ in L is convergent to x in L if, and only if*

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } m \geq N \Rightarrow x_m \in B_{\{u_1, \dots, u_d\}}(x, \varepsilon).$$

Summarizing all these results, we have:

Theorem 2.6. *any finite dimensional P - nN space is a P - N space and it's topology agrees with that generated by the derived norm \mathcal{F}^∞ .*

Definition 2.7. A sequence $\{x_m\}$ in P - nN space (L, \mathcal{F}, τ) is cauchy sequence if $\lim_{m, r \rightarrow \infty} F_{x_m - x_r, y_2, \dots, y_{n-1}, y} = \mathcal{H}_0$, for every $y \in L$.

Definition 2.8. The probabilistic n -normed space (L, \mathcal{F}, τ) is a probabilistic n -Banach space if every cauchy sequence in L is convergent to some x in L .

Theorem 2.9. *A probabilistic n -normed space (L, \mathcal{F}, τ) is a probabilistic n -Banach space if, and only if (L, \mathcal{F}, τ) is a probabilistic Banach space.*

Proof. By lemma 2.3, the convergence in the probabilistic n -norm is equivalent to that in the derived norm, it suffices to show that $\{x_m\}$ is Cauchy with respect to the probabilistic n -norm if, and only if it is Cauchy with respect to the derived norm. But it is clear since $\{x_m\}$ is Cauchy with respect to the probabilistic n -norm if, and only if $\lim_{m,r \rightarrow \infty} F_{x_m - x_r, y_2, \dots, y_{n-1}, y} = \mathcal{H}_0$ for every $y_2, \dots, y_{n-1}, y \in L$, if and only if $\lim_{m,r \rightarrow \infty} F_{x_m - x_r, y_2, \dots, y_{n-1}, u_i} = \mathcal{H}_0$ for every $i = 1, \dots, d$, if, and only if

$$\lim_{m,r \rightarrow \infty} F_{x_m - x_r}^\infty = \mathcal{H}_0$$

if, and only if $\{x_m\}$ is Cauchy with respect to derived norm. \square

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